J. Fluid Mech. (1966), *vol.* 25, *part* 4, *pp.* 769–786
Printed in Great Britain

The effect of longitudinal viscosity on the flow at a nozzle throat

By M. SICHEL
Department of Aerospace Engineering,
University of Michigan, Ann Arbor

(Received 21 July 1965 and in revised form 23 November 1965)

An inviscid transonic theory appears to be inadequate to describe the flow near the throat of a converging–diverging nozzle during the transition from the symmetrical Taylor (1930) type of flow to the subsonic–supersonic Meyer (1908) flow. A viscous transonic equation taking account of heat conduction and longitudinal viscosity has been developed previously (Cole 1949; Sichel 1963; Szaniawski 1963). An exact, nozzle-type of similarity solution of the viscous transonic equation, similar to the inviscid solution of Tomotika & Tamada (1950), has been found. This solution does provide a description of the gradual transition from the Taylor to the Meyer flow and shows the initial stages in the development of a shock wave downstream of the nozzle throat. The solution provides a viscous, shock-like transition from an inviscid, supersonic, accelerating flow to an inviscid, subsonic, decelerating flow.

1. Introduction

As the back pressure decreases, the flow through a converging–diverging nozzle changes from one which is symmetrical with respect to the throat to an asymmetrical flow with subsonic flow upstream and supersonic flow downstream of the throat. The two types of flow are illustrated in figure 1 (*b*). The transition between these two classes of nozzle flow has formed the subject of many investigations. Many important features of such transitional flows are adequately explained by a simple one-dimensional or hydraulic theory with normal shocks in the supersonic portion of the nozzle located to satisfy the downstream boundary condition on pressure. However, to resolve the details of the flow near the nozzle throat, which is intimately related to the nozzle-wall curvature, solutions of the two-dimensional or axisymmetric gasdynamic equations must be investigated. The initial phases of the transition during which shock-wave formation first starts in the neighbourhood of the throat are of particular interest; however, the results of a number of investigators indicate that the inviscid equations alone are unable to provide an adequate explanation of such a transitional flow. It appears that, to explain properly the initial stages in the shock formation, or development of shock-wave structure near the throat, at least the effect of longitudinal viscosity must be included in the conservation equations. Development of a viscous theory of such transitional flows forms the subject of the present paper.

The two-dimensional asymmetrical flow at a nozzle throat was first calculated by Meyer (1908) using a truncated series solution of the exact potential equation of an inviscid perfect gas. The calculations are straightforward and the solution appears to give a quite reasonable description of subsonic–supersonic nozzle flow. A similar series approach was applied by Taylor (1930) to flow near the throat of a two-dimensional symmetrical nozzle. As the maximum velocity on the nozzle axis increased, Taylor's calculations showed the development of pockets of supersonic flow near the nozzle surface. Taylor, however, found that, carrying terms up to the fourth degree in the double series in x and y, symmetrical solutions no longer exist when the peak velocity on the nozzle axis exceeds some maximum value. For a ratio $h/R = \frac{1}{4}$, where h is the half height of the nozzle and R the radius of curvature of the wall, there are no solutions for maximum velocities exceeding $0{\cdot}855a$, where a is the speed of sound.

Görtler (1939) showed that the series employed by Taylor tends to diverge as the velocity near the throat approaches sonic velocity, and suggested that the difficulty in Taylor's solution may be due to the neglect of higher-order terms which are cut off by the truncation process. Görtler (1939) attempted to extend Taylor's solution to the case of transitional flow by relaxing the requirement of symmetry with respect to the nozzle throat; however, a number of artificial assumptions regarding the series coefficients were required, making the convergence of his solution suspect.

Emmons (1946) used the method of relaxation to obtain numerical nozzle-flow solutions of the inviscid gasdynamic equations. Emmons (1946) postulated that the transition from the symmetrical Taylor to the asymmetrical Meyer type of nozzle flow starts with the formation of shock waves within the pockets of supersonic flow near the wall. This postulate was borne out by the calculations. Below a peak centreline Mach number M of 0·812 the compressible solutions were very much like the flow through a venturi. However, as the maximum centreline Mach number increased beyond this value, shock waves had to be placed in the pockets of supersonic flow in order to eliminate residuals in the relaxation calculations, and for sufficiently large M the shock waves within the two supersonic pockets joined at the nozzle centre. On the other hand, the numerical results contained several inconsistencies. The appearance of shock waves is sudden; that is, rather than gradually growing outward from some point in the flow the shock wave, when it first appears, is of finite length. A second difficulty is that there is a discontinuous rise in velocity immediately behind the shock waves. Emmons (1946) points out that this effect is caused by a discontinuity in the streamline curvature which occurs when a weak normal shock wave is adjacent to a curved wall. A similar effect was observed experimentally by Ackeret, Feldman & Rott (1946) and has also been discussed by Oswatitsch & Zierep (1960) and by Pearcy (1962). Since gradients behind weak shock waves adjacent to curved surfaces must be of the same order as gradients within the shock structure, the assumptions which permit the use of Hugoniot jump conditions across the shock are clearly violated. As Emmons (1946) has observed, a perfect-fluid theory including shock discontinuities across which the Hugoniot conditions hold appears to be inadequate to describe the nature of the transitional flow when shock waves first appear within the nozzle, rather a theory which includes the effects of fluid viscosity is required.

All the solutions described above are in some sense approximate solutions of the inviscid gasdynamic equations. Tomotika & Tamada (1950), on the other hand, found a mathematically exact nozzle-type solution of the transonic equation, which is an approximate equation valid only for regions of inviscid flow with Mach numbers near one. This approach avoids questions of the convergence of the series or of the numerical methods. Tomotika & Tamada (1950) obtained an exact similarity solution of the transonic equation describing both the Taylor and Meyer type of flow; however, they concluded that the flow of Meyer's type cannot be approached in a continuous manner from the group of solutions for the flow of Taylor's type, at least on the basis of the inviscid equations.

It appears that an adequate explanation of the transition from the Taylor- to the Meyer-type flow requires consideration of an equation which includes viscous terms such that the formation of shock waves is inherent in the equations themselves. It has been shown (Cole 1949; Sichel 1963; Szaniawski 1963; Ryzhov & Shefter 1964) that, in regions of transonic flow in which the longitudinal or compressive viscosity is dominant such as in the interior of a weak shock, the flow can be described by an equation which is identical with the transonic equation except for an additional viscous term, and which has sometimes been called the viscous transonic or V–T equation. A nozzle-type similarity solution, similar to that of Tomotika & Tamada (1950), has been found for the V–T equation and appears to provide a reasonable picture of the gradual transition from the Taylor to the Meyer type of flow. This viscous transonic solution forms the subject of this paper; however, because of the close relation to the work of Tomotika & Tamada (1950) their solution will first be discussed in detail.

2. The solution of Tomotika and Tamada

The approximate equations for inviscid two-dimensional transonic flow, which have for example been derived by Guderley (1962), can be written in the form

$$U_{YY} - (U^2)_{XX} = 0, \tag{1}$$

where

$$\epsilon U = (\bar{u}/a^*) - 1,$$

$$X = \tfrac{1}{2}(\gamma+1)\,\bar{x}/\lambda, \quad Y = ([\gamma+1]/2)^{\frac{1}{2}}\,\epsilon^{\frac{1}{2}}\bar{y}/\lambda,$$

with a^* the critical speed of sound, λ a characteristic dimension of the flow, and ϵ a small parameter proportional to the deviation of $\bar{u}/a^*$ from unity. In the above equations barred quantities are dimensional. Upon introducing the transformation

$$\left.\begin{aligned} U &= Z(S) + 2\sigma^2 Y^2, \\ S &= X + \sigma Y^2, \end{aligned}\right\} \tag{2}$$

Tomotika & Tamada (1950) found that equation (1) collapses to the non-linear ordinary differential equation

$$ZZ'' + (Z' - 2\sigma)(Z' + \sigma) = 0. \tag{3}$$

Since the flow described by equations (2) is symmetrical with respect to the X-axis, it can be considered to represent a nozzle flow. Tomotika & Tamada (1950) obtained the implicit analytical solution

$$(Z-2\sigma S)^2(Z+\sigma S) = 2\alpha^3/\sigma^3 \tag{4}$$

for equation (3), where the constant of integration α determines the nature of the solution. The arbitrary constant σ, for which Tomotika & Tamada originally used the value 1·0, determines the slope of the two special solutions $Z = 2\sigma S$, and $Z = -\sigma S$ corresponding to $\alpha = 0$. The more general transformation above was introduced in a later paper by Tomotika & Hasimoto (1950).

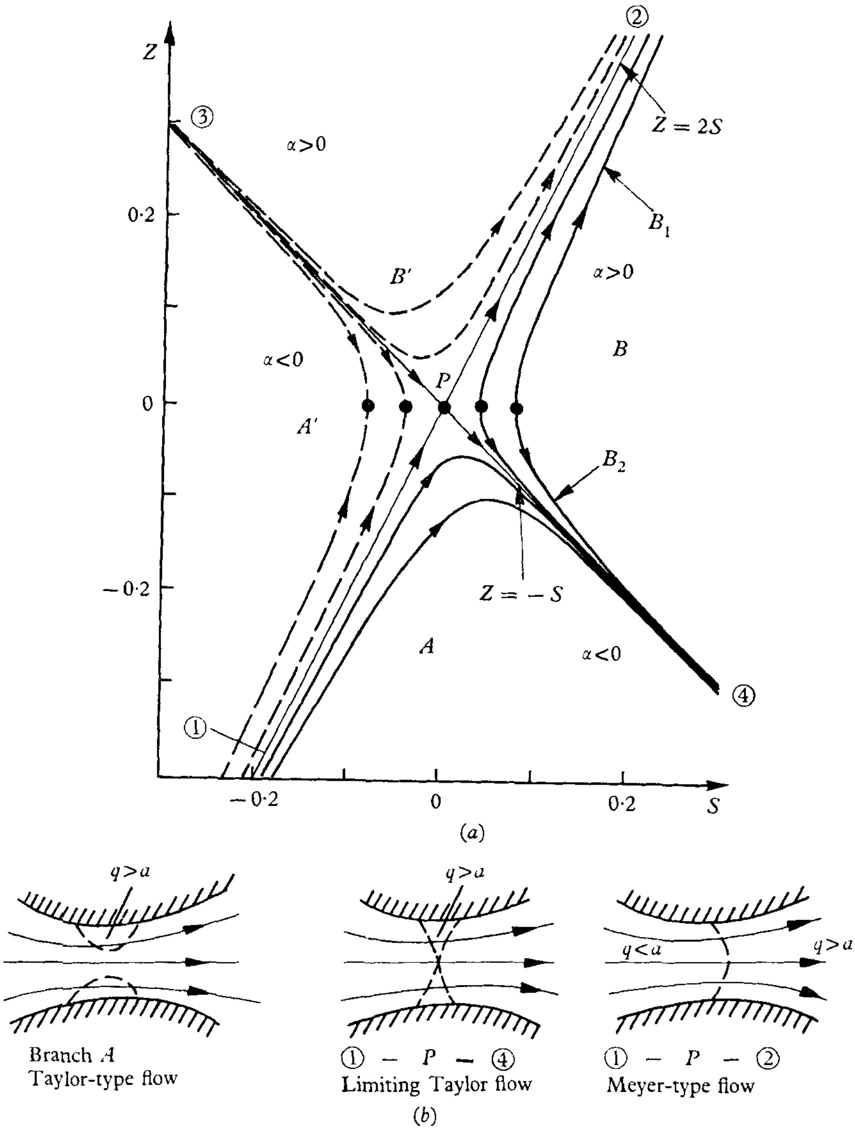


FIGURE 1. (*a*) Z versus S from the solution of Tomotika & Tamada (1950). (*b*) Nozzle flows corresponding to the solution of Tomotika & Tamada (1950).

Using the condition of irrotationality, Tomotika & Tamada were able to compute the $\bar{y}$-velocity component and hence to determine the streamlines of the flow.

The behaviour of the function $Z(S)$, which is equal to the velocity U on the nozzle axis, is reproduced in figure 1 (a) for various values of α, and for $\sigma = 1{\cdot}0$. The solution curves have four branches separated by the special solution curves $Z = 2\sigma S$ and $Z = -\sigma S$ corresponding to $\alpha = 0$. Figure 1(b) shows that nozzle

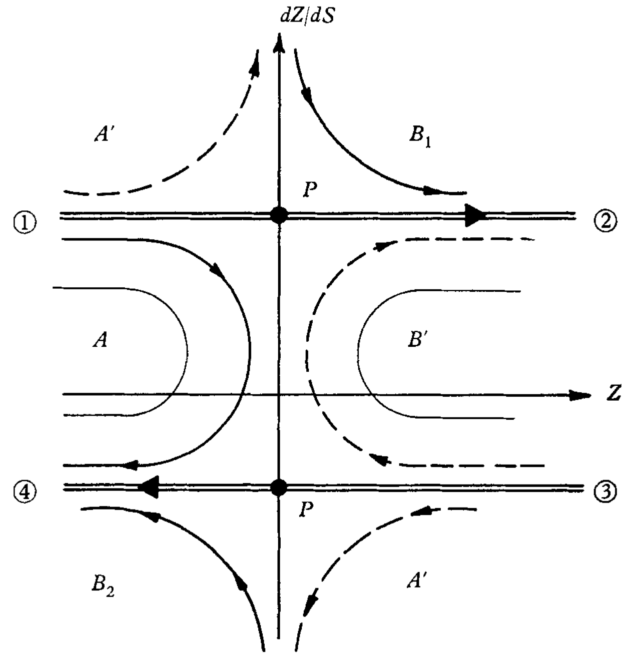


FIGURE 2. Phase-plane behaviour of the Tomotika & Tamada solution.

flows constructed from branch A solutions correspond to the Taylor-type, symmetrical, nozzle flow. As $\alpha \to 0$, curves of branch A asymptotically approach (1)–P–(4), which has a discontinuous slope at the sonic point P and represents the limiting Taylor flow with the maximum velocity on the nozzle axis just sonic. The special solution $Z = 2\sigma S$ yields the Meyer-type asymmetrical flow, as shown in figure 1(b), and, as shown later, is identical with the first few terms of the Meyer solution. Branches A' and B have infinite slope at the sonic point, and so are not physically meaningful, while branch B' is entirely supersonic and so is not of interest here. The sonic point $Z = 0$ is a singularity, for it is clear from equation (3) and figure 1(a) that only the singular solutions with $Z' = 2\sigma$ or $Z' = -\sigma$ can pass through the sonic point with finite Z'' or curvature. It should be remarked that the direction of the flow as indicated by the arrows in figure 1 (a) and (b) seems to have been reversed in the original paper.

Tomotika & Tamada (1950) suggested that the limiting Taylor flow (1)–P–(4) will change discontinuously to the Meyer type (1)–P–(2) provided the nozzle-exit conditions change sufficiently. However, their solution does not permit a continuous transition from the limiting Taylor solution to the Meyer solution, as becomes particularly apparent when the solution is plotted in the phase plane (see figure 2). In this plane the sonic line $Z = 0$ acts as a barrier such that subsonic solutions can never become supersonic and vice versa except for the two singular solutions $Z = 2\sigma S$ and $Z = -\sigma S$. The question now to be examined is whether taking account of the viscosity in the formulation of the flow equations can resolve this difficulty.

3. Viscous transonic nozzle solution

Within the structure of shock waves the terms of the Navier–Stokes equation due to compressive or longitudinal viscosity, and due to heat conduction, are of the same order of magnitude as the non-linear convective terms, for it is the balance between the steepening convective terms and the smoothing dissipative terms which leads to the existence of steady-state shock-wave structures. One-dimensional Navier–Stokes solutions of shock-wave structure are well known (Hayes 1958); however, there are regions of flow, which might aptly be called thick shock waves, where the main effect is still a balance between convection and dissipation but where the flow is not necessarily one-dimensional.

In the transonic case, approximate equations describing the flow within such a thick shock layer have been derived from the full Navier–Stokes equations (Sichel 1963; Szaniawski 1963) by using an expansion in the small parameter ϵ coupled with stretching of the co-ordinates. The resultant first-order equations in normalized form are as follows:

$$U_{XX} - 2UU_X + V_Y = 0, \tag{5}$$

$$U_Y = V_X. \tag{6}$$

The dimensionless co-ordinates X, Y are in this case related to the physical co-ordinates $\bar{x}, \bar{y}$ by

$$X = A(\bar{x}/\eta), \quad Y = A\Gamma^{\frac{1}{2}}\epsilon^{\frac{1}{2}}(\bar{y}/\eta), \tag{7}$$

where $$A = \Gamma\{1+(\gamma-1)/P_r''\}^{-1}, \quad \Gamma = (1/a)\,(\partial\rho a/\partial\rho)_s,$$

and for a perfect gas $\Gamma = \frac{1}{2}(\gamma+1)$. P_r'' is the Prandtl number based on the compressive viscosity μ'', which is related to the bulk and shear viscosities μ' and μ by

$$\mu'' = \tfrac{4}{3}\mu + \mu',$$

and is also sometimes called the longitudinal viscosity (Hayes 1958). The quantity η is given by

$$\eta = \mu^{*\prime\prime}/\epsilon\rho^* a^* = \nu^{*\prime\prime}/\epsilon a^*,$$

and is of the same order of magnitude as the thickness of a weak shock wave (Lighthill 1956) with upstream critical Mach number $M_1^* = 1+\epsilon$. Thus the characteristic dimension used to normalize the co-ordinates is of the order of the thickness of a weak shock wave. U and V are related to the actual $\bar{x}$ and $\bar{y}$ components of velocity by

$$\bar{u}/a^* = 1+\epsilon U, \quad \bar{v}/a^* = \epsilon^{\frac{3}{2}}\Gamma^{\frac{1}{2}}V. \tag{8}$$

Equation (8) expresses the well-known fact that in transonic flow, when

$$(\bar{u}/a^*)-1 \sim O(\epsilon), \quad \text{then} \quad (\bar{v}/a^*) \sim O(\epsilon^{\frac{3}{2}}).$$

Using the condition of irrotationality (6) to eliminate V from equation (5) now yields the following equation for U:

$$U_{XXX} - (U^2)_{XX} + U_{YY} = 0. \tag{9}$$

Except for the viscous U_{XXX} term (9) is identical with equation (1), the inviscid transonic equation expressed in terms of U.

Again using the transformation (2), equation (9) for U reduces to the ordinary differential equation

$$Z''' - 2ZZ'' - 2(Z' - 2\sigma)(Z' + \sigma) = 0. \tag{10}$$

Thus Tomotika & Tamada's similarity transformation also works for the viscous transonic equation. The resultant solution again represents a nozzle-type flow symmetrical with respect to the X-axis. As before, the function $Z(S)$ represents the centreline velocity distribution. Equation (10) is identical with (3) except for the viscous Z''' term.

From equation (10) it is evident that the special inviscid solutions

$$Z = 2\sigma S, \quad Z = -\sigma S \tag{11}$$

are also solutions in the viscous case. With the presence of the viscous Z''' term in (10), the sonic point $Z = 0$ is no longer a singularity so that solution curves passing through the sonic point are not restricted to the two special solutions (11). So far it has not been possible to obtain any other analytical solutions of equation (10); however, it should be possible to obtain solutions $Z(S)$ numerically.

Equation (10) is such that, for finite Z, Z' and Z'', choice of initial conditions $Z(S_0)$, $Z'(S_0)$, and $Z''(S_0)$ at some point S_0 will determine a unique solution (Coddington & Levinson 1955); however, the question of what initial conditions to choose is certainly not a trivial one. In what might be termed the direct nozzle problem specification of the nozzle contour and conditions upstream and downstream of the throat lead to a boundary-value problem for the viscous transonic equation (9). Sichel (1963) has discussed the question of properly set boundary conditions and given a uniqueness proof for the viscous transonic equation valid for subsonic flows while Szaniawski (1964*a*, *b*) and Kopystynski & Szaniawski (1965) have investigated the direct viscous transonic nozzle problem using series-expansion methods. The present problem, on the other hand, is indirect in that the question asked is whether any of the flow fields corresponding to solutions $Z(S)$ of equation (10) satisfy boundary conditions representative of flow through a nozzle throat, while also representing the transition from the Taylor to the Meyer type of flow. In some similarity analyses, such as the Blasius-flat-plate-boundary-layer solution, the boundary conditions which the ordinary differential equation obtained from the partial differential equation must satisfy are precisely specified ; however, this is not the case here. All that is known is that the transitional solutions being sought should start where $Z(S) < 0$ (subsonic flow) and $Z'(S) > 0$ (velocity increasing), must pass through a maximum which may be either subsonic or supersonic, and then must decrease; however, at this point it is not even known whether such solutions of equation (10) exist. Consequently the general properties of equation (10) must first be studied to provide a guide for the evaluation of numerical solutions. For this purpose the qualitative behaviour of solution trajectories in the phase space will be investigated.

Since equation (10) is of third order, it becomes necessary to consider the behaviour of solution trajectories in the three-dimensional Z, Z', Z'' phase space, a more difficult problem than the more usual phase-plane analysis of second-order systems. Although equation (10) can be integrated once to yield the second-order equation

$$Z'' - 2ZZ' + 2\sigma Z + 4\sigma^2 S = C_1 \tag{12}$$

with C_1 a constant of integration, equation (12) contains the independent variable S, so that it is no longer autonomous, and a separate phase plane is needed for each S. Hence, it is more straightforward to deal with the original third-order equation, and with the three-dimensional phase space.

Letting $p = Z'$, $q = p' = Z''$, the solution trajectories in the (Z, p, q)-space satisfy the equations

$$\frac{dZ}{p} = \frac{dp}{q} = \frac{dq}{2\{(p-2\sigma)(p+\sigma)+Zq\}}. \tag{13}$$

There are no singular points where $dp = dq = dZ = 0$ simultaneously; however, the trajectories $p = 2\sigma$, $q = 0$ and $p = -\sigma$, $q = 0$ corresponding to the two inviscid solutions $Z = 2\sigma S$ and $Z = -\sigma S$ are singular lines in the sense that $dp = dq = 0$ on each of them.

A composite picture of the phase space behaviour can be gained by studying the two-dimensional trajectories obtained when Z in equation (13) is held constant. These curves in the planes Z = const. are not solution trajectories but are tangent to the projections of these trajectories at the point where they cross the Z = const. plane. In these (q, p)-planes the points $(0, -\sigma)$ and $(0, 2\sigma)$ are now singularities where $dp = dq = 0$. Using well-established methods for studying the singularities of second-order systems (Minorsky 1962), it has been shown (Sichel 1965) that the point $(0, 2\sigma)$ behaves as a saddle point for all values of Z; however, the directions of the two separatrixes of the saddle point do vary with Z. The singularity $(0, -\sigma)$, on the other hand, changes in character with Z from an unstable node to an unstable focus to a stable focus and finally to a stable node corresponding respectively to the ranges

$$Z > \sqrt{(6\sigma)}, \quad \sqrt{(6\sigma)} > Z > 0, \quad 0 > Z > -\sqrt{(6\sigma)}, \quad \text{and} \quad Z < -\sqrt{(6\sigma)}.$$

A typical set of such trajectories, plotted by the method of isoclines, are shown in figure 3 for the particular value $Z = -1$. The parabolas in figure 3 are lines of constant slope.

In assessing the significance of the above results, it is extremely important to recognize that these 'crossing trajectories' are not solution trajectories, and that there are no true singularities in the phase space as for example in the case of the one-dimensional shock wave (Ludford 1951) or detonation structure (Wood 1961). The phase-space behaviour is, however, largely determined by the behaviour of the solution trajectories near the singular solutions $Z = 2\sigma S$, $Z = -\sigma S$. Solutions starting near the $Z = 2\sigma S$ trajectory, no matter how close, will ultimately deviate from this trajectory as S increases. From the crossing-trajectory diagrams and the behaviour of the crossing trajectories near the singularity at $q = 0$, $p = -\sigma$ it appears that there may be solutions starting infinitesimally near $Z = 2\sigma S$ and with $p < 2\sigma$ and $q < 0$, which will pass through a maximum in Z on the plane $p = 0$ and will then, as Z decreases, asymptotically approach the solution $Z = -\sigma S$. Numerical integration of equation (10) indicates that such solutions do indeed exist.

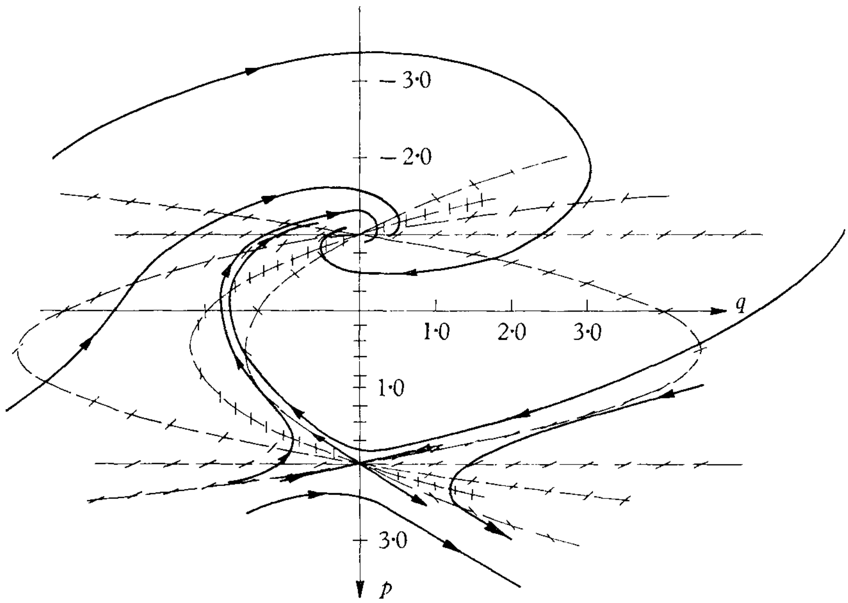


FIGURE 3. Crossing trajectories for $Z = -1{\cdot}0$.

Figure 4 shows a set of numerical solutions $Z(S)$ obtained by starting the integration very close to the singular solution $Z = 2\sigma S$ for different initial values of Z. Starting from an essentially subsonic velocity profile on the centreline this sequence of solutions shows the gradual development of what appears to be a shock wave, and these are exactly the type of transitional solutions being sought.

For the starting-points in the numerical calculations Z_0' was chosen very close to 2·0 for different values of Z_0. Z_0'' was adjusted to make the integration constant $C_1 = 0$ for then, as can be seen from equation (12), the phase of $Z(S)$ will be such that the solutions will be asymptotic to $Z = 2\sigma S$ and $Z = -\sigma S$. The starting values used in calculating the curves in figure 4 are given in table 1.

The (q, p)-plane projections of trajectories corresponding to curves A and C of figure 4 are shown in figure 5 and support the results of the 'crossing-trajectory', singular-point analysis. Because of the unstable nature of the special solution $p = 2\sigma$, $q = 0$, numerical solutions, though started very close to this solution, do not in general correspond exactly to one of the solutions which asymptotically approaches $Z = 2\sigma S$, as is evident from the special plots showing the detailed behaviour near the two inviscid solutions. The situation is similar to that encountered in plane-shock-structure problems, where numerical integration

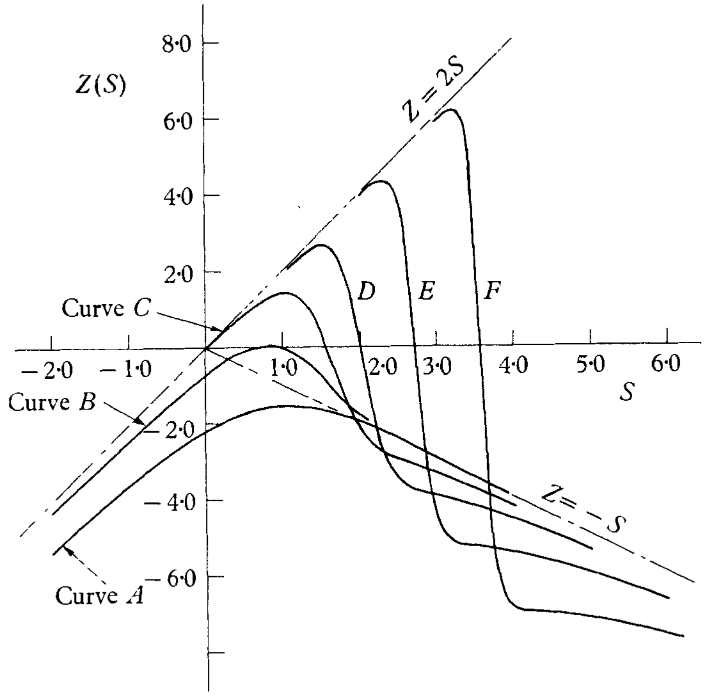


FIGURE 4. Numerical solutions of the equation $Z''' - 2(ZZ')' + 2\sigma Z' + 4\sigma^2 = 0$ for $\sigma = 1{\cdot}0$. The solutions are asymptotic to $z = 2S$ and $z = -S$.

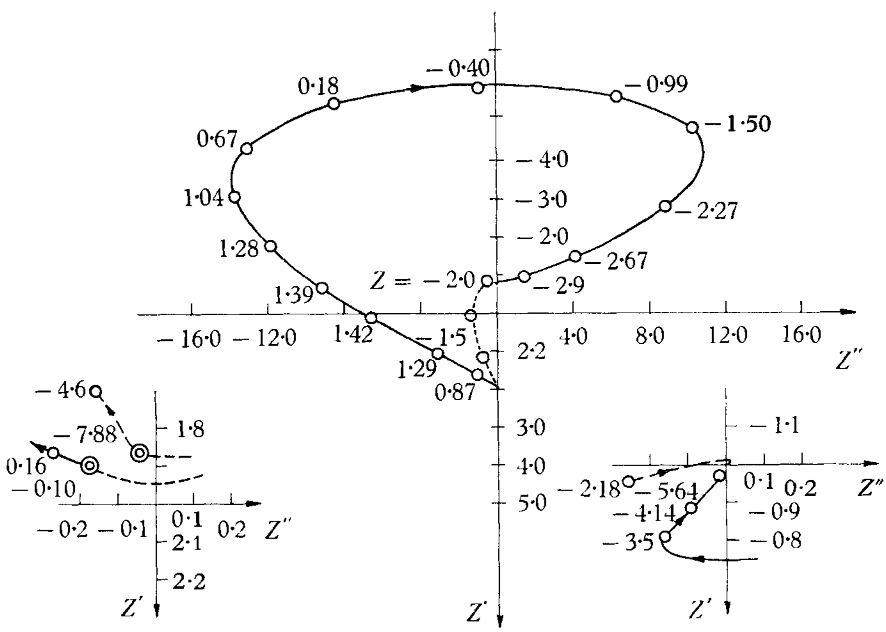


FIGURE 5. Projection of the solution trajectories in the (Z', Z'')-plane, showing also the detailed behaviour near $Z'' = 0$, $Z' = 2{\cdot}0$ and near $Z'' = 0$, $Z' = -1$. Numbers near circled points indicate corresponding values of Z. ———, Subsonic–supersonic; – – –, subsonic; ◎, starting value.

must be started near the downstream saddle point. In figure 5 the points preceding the starting values were obtained by backward integration. A Runge–Kutta fourth-order method was used to integrate the equation.

As the maximum supersonic value of $Z(S)$ increases, it can be seen from figure 4 that the slope $Z'(S)$ in the transition region becomes progressively steeper. If the dimensionless velocity upstream of a weak normal shock is $1+\epsilon U_1$, then the downstream velocity will be $1-\epsilon U_1$ provided the Hugoniot conditions hold. In figure 4 the downstream velocity at first overshoots the Hugoniot value; however, as $Z_{\max}$ increases the jump conditions more closely approach those of a normal shock. As $Z_{\max}$ increases the large values of Z'' and Z' in the transition region make the terms Z'' and ZZ' dominant in equation (12); however, the equation

$$Z''-2ZZ' = 0$$

Curve ...	A	B	C	D	E	F
Z_0	$-7{\cdot}88$	$-3{\cdot}59$	$-0{\cdot}10$	$1{\cdot}90$	$3{\cdot}90$	$5{\cdot}90$
Z_0'	$1{\cdot}87$	$1{\cdot}901$	$1{\cdot}90$	$1{\cdot}90$	$1{\cdot}90$	$1{\cdot}90$
Z_0''	$-0{\cdot}0463$	$-0{\cdot}0695$	$-0{\cdot}180$	$-0{\cdot}580$	$-0{\cdot}980$	$-1{\cdot}380$

TABLE 1. Starting values for the singularity solution

formed by keeping these terms alone is just the one describing the Taylor (1910) structure of a weak shock wave. Thus, as $Z_{\max}$ increases, the supersonic–subsonic transition on the axis of the nozzle seems to approach the structure of a weak normal shock. These results further suggest that with $\sigma \ll 1$ solutions will be obtained such that there is essentially a weak normal shock near the nozzle axis which is modified by non-Hugoniot effects only for sufficiently large Y.

4. Construction of flow fields

A complete evaluation of the similarity solution described above requires the computation of the corresponding nozzle flow fields. For this purpose isotachs and streamlines must be determined and it is also necessary to relate the dimensionless solution in the (X, Y)-plane to the physical $(\bar{x}, \bar{y})$-plane.

Since the dimensionless speed $q = \bar{q}/a^*$ is given by

$$q = \{(1+\epsilon U)^2+\epsilon^3 V^2\Gamma\}^{\frac{1}{2}} = 1+\epsilon U+O(\epsilon^2), \tag{14}$$

it follows that isotachs correspond to contours of constant U to the present order of approximation. The streamline slope $(d\bar{y}/d\bar{x})_s$ is given by

$$(d\bar{y}/d\bar{x})_s = (\bar{v}/\bar{u}) = \epsilon^{\frac{3}{2}}\Gamma^{\frac{1}{2}}V+O(\epsilon^{\frac{5}{2}}), \tag{15}$$

or in terms of the stretched co-ordinates X and Y

$$(dY/dX)_s = \epsilon^2\Gamma V. \tag{16}$$

From the condition of irrotationality (6) and the similarity transformation (2) it follows that

$$V = 2\sigma YZ+4\sigma^2XY+f(Y), \tag{17}$$

where $f(Y)$ is a function of Y. From the V–T equation (5), and equation (12) for Z, it then follows that

$$f(Y) = \tfrac{4}{3}\sigma^3 Y^3 - C_1 Y + C_2. \tag{18}$$

Since $V(X, 0) = 0$ for nozzle flow, $C_2 = 0$. The constant C_1 depends upon the initial conditions used in evaluating Z and upon the origin S_0 chosen for S, and from equation (12) is readily seen to have the value

$$C_1 = Z''(S_0) - 2Z(S_0)\,Z'(S_0) + 2\sigma Z(S_0) + 4\sigma^2 S_0. \tag{19}$$

C_1 merely determines the phase of the solution $Z(S)$ with respect to the S co-ordinate. It follows that

$$V = 2Y(\sigma Z + 2\sigma^2 X + \tfrac{2}{3}\sigma^3 Y^2 - \tfrac{1}{2}C_1), \tag{20}$$

which is identical with the result of Tomotika & Tamada (1950) except for the constant C_1, which they set equal to zero but which has been retained here. Streamlines can now be determined by integrating equation (16) for different initial conditions, using V as given by (20).

Any streamline can be considered as the wall of a nozzle; however, it is of particular interest to choose a streamline with a predetermined ratio of nozzle half height to radius of curvature at the throat in order to compare the viscous transonic results to the inviscid calculations of Taylor (1930) and Tomotika & Tamada (1950). On the nozzle wall $V = 0$ at the throat so that

$$\sigma Z(X_t + \sigma Y_t^2) + 2\sigma^2 X_t + \tfrac{2}{3}\sigma^3 Y_t^2 - \tfrac{1}{2}C_1 = 0, \tag{21}$$

where the subscript t refers to the throat co-ordinates. Letting h be the half height at the throat of the nozzle and R_t the radius of curvature of the wall, it follows from equations (16) and (20) that

$$h/R_t = h(d^2\bar{y}/d\bar{x}^2)_t = 2\epsilon Y_t^2\{\sigma Z'(X_t + \sigma Y_t^2) + 2\sigma^2\}. \tag{22}$$

Equations (21) and (22) are sufficient to determine the throat co-ordinates X_t, Y_t once h/R_t and ϵ are specified, and with X_t, Y_t known the wall streamline is obtained by integrating (16).

The definition of the parameter ϵ, which characterizes the maximum deviation of the fluid velocity from the sonic value a^*, is arbitrary, but usually related to the particular problem under investigation. In the study of non-Hugoniot shock structure (Sichel 1963), it was convenient to let $\epsilon = (\bar{u}_1/a^*) - 1$, where $\bar{u}_1$ is the velocity of the undisturbed flow upstream of the shock wave, while in flow about bodies the choice $\epsilon = (M_\infty - 1)$, where M_∞ is the Mach number of the undisturbed flow, is frequently made. In the present case, since neither of the above definitions is suitable, ϵ represents the value of $(\bar{u}/a^*) - 1$ corresponding to points where $U = 1$. As a consequence of this definition only those nozzle solutions for which $U \sim O(1)$ are consistent with the expansion scheme used here.

It now is necessary to establish the connexion between the nozzle solution above, which is expressed in terms of the dimensionless co-ordinates X and Y, and the physical plane. The V–T equation in the dimensionless form (9) provides the basis of a viscous transonic similitude. For each solution of equation (9) there exists a family of physical flows corresponding to different values of the parameters ϵ, Γ, $\nu^{*''}$, a^* and Pr$''$. The members of this family are similar, and each point X, Y of the dimensionless solution defines a set of corresponding points in the family of similar solutions. This similitude is closely related to the more conventional transonic one, discussed, for example, by Guderley (1962); the main difference between the two being the nature of the characteristic dimension. In the conventional similitude the length λ in equation (1) is a characteristic dimension of the flow such as the chord length of an airfoil, for example; however, in the viscous transonic case the characteristic length η introduced in equation (7) is of the order of the thickness of a weak shock wave. With decreasing ϵ, λ will remain fixed however η must increase.

From equation (7) it follows that for given X and Y corresponding values of $\bar{x}$ and $\bar{y}$ are given by

$$\bar{x} = (\eta/A)\,X, \quad \bar{y} = (\eta/A\Gamma^{\frac{1}{2}}\epsilon^{\frac{1}{2}})\,Y, \tag{23}$$

so that for fixed fluid properties and a^*, $\bar{x} \sim \epsilon^{-1}$ and $\bar{y} \sim \epsilon^{-\frac{3}{2}}$ at corresponding points since $\eta = \nu^{*''}/\epsilon a^*$. The streamlines in the $(\bar{x}, \bar{y})$-plane corresponding to the streamlines passing through a particular point X_1, Y_1 in the (X, Y)-plane will be called corresponding streamlines. The reference point $\bar{x}_1, \bar{y}_1$ transforms according to (23), but since $v \sim O(\epsilon^{\frac{3}{2}})$ the streamline slope, $d\bar{y}/d\bar{x}$, must also be $O(\epsilon^{\frac{3}{2}})$. On the other hand, if all points on corresponding streamlines transformed according to (23) the result would be $(dy/dx) \sim O(\epsilon^{-\frac{1}{2}})$. This strange behaviour, which was also noted by Guderley (1962), is responsible for the appearance of ϵ^2 in equation (16) for streamlines in the (X, Y)-plane. As a consequence, even though Z, U, and V are functions only of X and Y, the streamlines in the (X, Y)-plane will vary with the parameter ϵ.

The relation between the dimensionless and physical nozzle solutions is now established. In the $\bar{x}$-direction the length of the region of interest will be $O(\eta)$. Since $\nu^{*''}/a^*$ is of the order of a mean free path, $\eta = \nu^{*''}/a^*\epsilon$ will be very small unless the density is low or ϵ is very small. The half height h is the other significant dimension of the flow. From equation (22) it follows that streamlines with h/R_t fixed will not at the same time be corresponding streamlines. Thus assuming that $\{\sigma Z'(X_t+\sigma Y_t^2)+2\sigma^2\} \sim O(1)$ it follows from (22) that

$$y_t \sim O\{(h/R_t)^{\frac{1}{2}}\,(\nu^{*''}/a^*\epsilon^2)\}, \tag{24}$$

and, if $(h/R_t)^{\frac{1}{2}} \sim O(1)$, it follows that

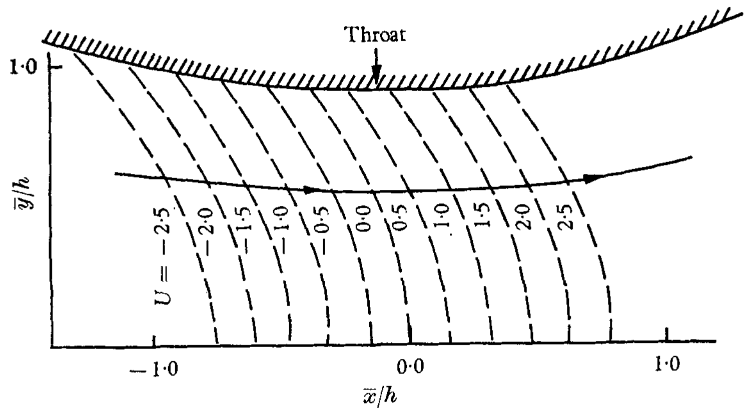
$$h/\eta \sim (1/\epsilon), \tag{25}$$

so that the half height will be much greater than the thickness of a weak shock provided that $\epsilon \ll 1$.

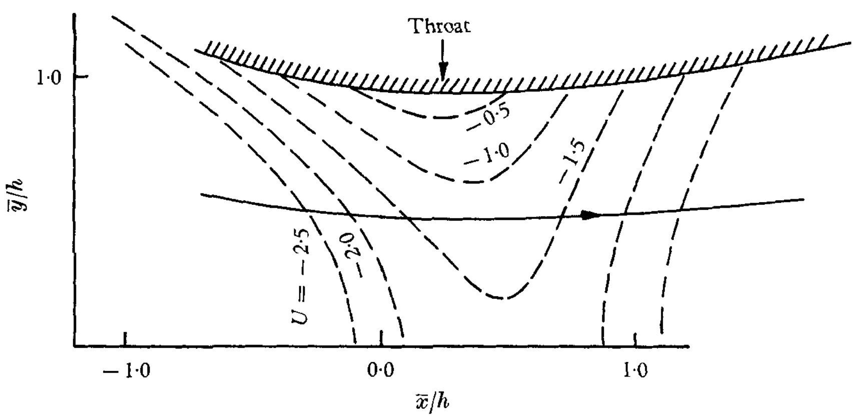
For the particularly simple inviscid solution, $Z = 2\sigma(S-S_0)$, explicit expressions for the isotachs, the vertical velocity, and the streamlines can be found, and it is readily shown (Sichel 1965) that in terms of the physically more meaningful dimensionless co-ordinates $\bar{x}/h$ and $\bar{y}/h$

$$\left.\begin{aligned} \frac{\bar{u}}{a^*}-1 &= \epsilon U = \left(\frac{\beta}{2\Gamma}\right)^{\frac{1}{2}}\left(\frac{\bar{x}}{h}-\frac{\bar{x}_0}{h}\right)+\tfrac{1}{2}\beta\left(\frac{\bar{y}}{h}\right)^2, \\ \frac{\bar{v}}{a^*} &= \epsilon^{\frac{3}{2}}\Gamma^{\frac{1}{2}}V = \frac{\bar{y}}{h}\left\{\beta\left(\frac{\bar{x}}{h}-\frac{\bar{x}_0}{h}\right)+\frac{\sqrt{(2\Gamma)}\,\beta^{\frac{3}{2}}}{6}\left(\frac{\bar{y}}{h}\right)^2\right\}. \end{aligned}\right\} \tag{26}$$

In the case of a perfect gas with $\Gamma = \frac{1}{2}(\gamma+1)$ equation (26) is identical with the nozzle velocity distribution obtained from the first three terms of Meyer's double expansion for the velocity potential (Meyer 1908; Hall & Sutton 1962), except that in accordance with the discussion above the choice of the characteristic



(a)



(b)

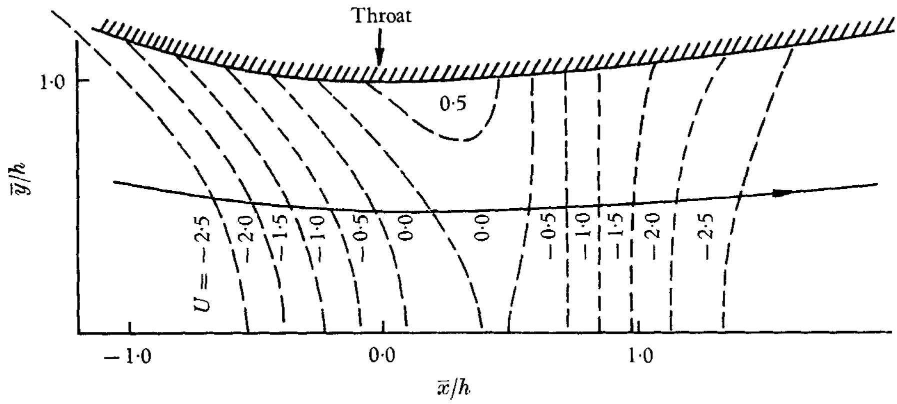
FIGURE 6. For legend see facing page.

nozzle dimension is no longer arbitrary. In equation (26) $\beta = h/R_t$, and from equations (22) and (23) it is readily shown that the nozzle half height h is related to η by

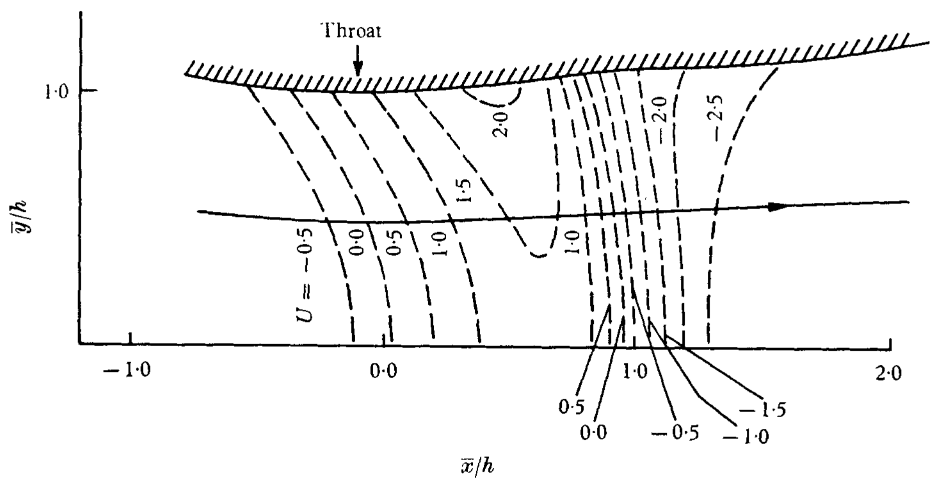
$$\frac{h}{\eta} = \frac{\beta^{\frac{1}{2}}}{2^{\frac{3}{2}}\Gamma^{\frac{1}{2}}\epsilon\sigma A}.$$

The point $\bar{x}_0/h$ is the location of the sonic point on the nozzle axis. Integration of equation (16) for the streamlines is straightforward and will not be reproduced here. Isotachs and streamlines corresponding to the Meyer-type flow with $\sigma = 1$ are shown in figure 6 (a) plotted in the $(\bar{x}/h, \bar{y}/h)$-plane. The wall streamline has been chosen so that $h/R_t = 0{\cdot}25$ as in the calculations of Taylor and of Tomotika & Tamada.

In the more general case streamlines can only be obtained by numerical integration of equation (16), and it is no longer possible to obtain simple expressions for U and V in terms of $\bar{x}/h$ and $\bar{y}/h$. Figures 6(*b*), (*c*), and (*d*) show the streamlines and isotachs corresponding to solution curves A, B, and C in figure 4.



(*c*)



(*d*)

FIGURE 6. Isotachs and streamlines in nozzles for $\epsilon = 0{\cdot}1$, $\bar{u}/a^* = 1+\epsilon U$, $\gamma = 1{\cdot}4$, $\beta = h/R_t = 0{\cdot}25$. (*a*) $Z = 2S$ (Meyer's solution), (*b*) curve A of figure 4, (*c*) curve B of figure 4, (*d*) curve C of figure 4.

Figure 6 (*b*) represents a Taylor-type nozzle flow with subsonic velocities throughout but regions of high-velocity flow near the nozzle wall. In figure 6(*c*) the maximum centreline velocity is just sonic, while there are pockets of supersonic flow near the nozzle wall. Upstream of the throat the flow is similar to the Meyer-type flow of figure 6(*a*), but downstream a certain crowding of the isotachs as compared with the Taylor flow of figure 6(*b*) is evident. In figure 6(*d*) the velocity along the axis becomes supersonic beyond the throat but this supersonic region is followed by a rapid deceleration to subsonic flow. Figure 6(*d*) appears to indicate an initial stage in the development of a shock wave downstream of the throat. There are undulations in the portion of the nozzle wall downstream of the throat, which follow from the rapid changes in $Z'(S)$. This boundary condition goes with the similarity solution for, as mentioned previously, there is no freedom to choose the streamline shape in the present case. Nevertheless, figure 6 (*d*) depicts flow through a nozzle with a throat or section of minimum area.

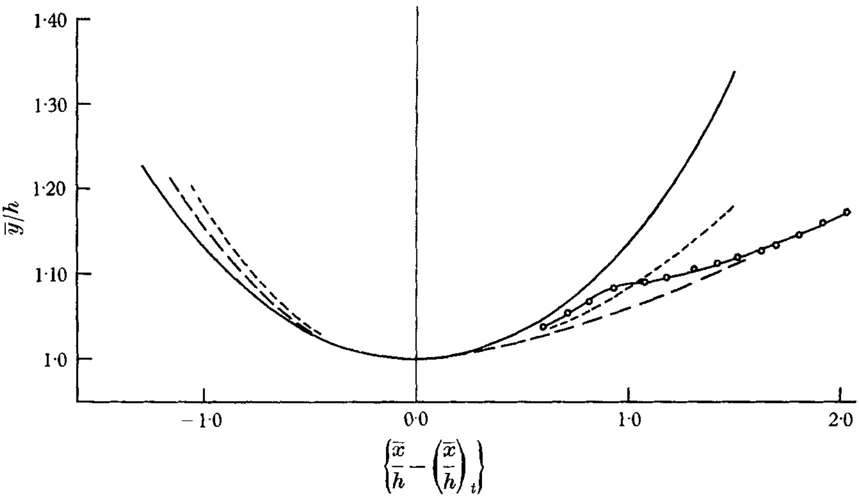


FIGURE 7. Detailed plot of nozzle contours for $\beta = 0·25$, $\epsilon = 0·1$, $\sigma = 1·0$. ———, Meyer solution; – – – –, curve A; — — —, curve B; —o—o—, curve C. The Meyer solution and curve C merge upstream.

The wall streamlines of figure 6 are shown in greater detail in figure 7. The nozzle contour of figure 6 (*d*) coincides with the Meyer-flow contour upstream of the throat while far downstream this contour (curve C) approaches the shallower Taylor-type contour. This result is not surprising since the flow in figure 6(*d*) does change from a Meyer to a Taylor type of flow downstream of the throat.

5. Discussion

By including the effect of longitudinal viscosity in the equation for plane transonic flow it has been possible to obtain solutions which provide a smooth transition from the Taylor- to the Meyer-type of nozzle flow, and which show what happens in the initial stages of shock formation downstream of the nozzle throat. The difficulties near the sonic point, so characteristic of the inviscid analysis, disappear when the nozzle problem is formulated in terms of the viscous transonic equation. While these solutions are exact, they are purchased at the penalty of not being able to specify an arbitrary nozzle-wall shape.

The solution obtained here is in some sense related to Taylor's weak-shock solution (Taylor 1910) describing the viscous transition between uniform, supersonic and subsonic flows. The viscous transonic solution yields a viscous transition between a supersonic flow of increasing and a subsonic flow of decreasing velocity. In each case viscous effects vanish upstream and downstream of the transition.

As the maximum value of the centreline velocity $U(X,0)$ increases the viscous transonic transition appears to approach the Taylor weak shock structure.

The horizontal scale of the portion of the nozzle flow under consideration here is of the order of η, the thickness of a weak shock wave, while the vertical scale h is of the order of η/ϵ. Unless $\epsilon \ll 1$, and the flow is one of low density, the throat width of the nozzles under consideration here will be extremely small. For example for nitrogen at 27·4 °C, with $p = 1$ atm, and $\epsilon = 0{\cdot}1$, $\eta \sim O(0{\cdot}001\,\mathrm{mm})$ while $h \sim O(0{\cdot}01\,\mathrm{mm})$. On the other hand, with $p = 0{\cdot}01$ atm, and $\epsilon = 0{\cdot}01$, $\eta \simeq O(1{\cdot}0\,\mathrm{mm})$ while $h \sim O(100\,\mathrm{mm})$, which is certainly of a more reasonable magnitude.

It will be difficult to obtain a precise experimental verification of the results obtained here. The presence of the nozzle-wall boundary layer will make it difficult to reproduce bounding streamlines or nozzle contours which agree exactly with the streamlines obtained from the similarity solution, and slight shifts of the boundary can cause relatively large changes in transonic flow. The region of interest will be extremely small unless the density is low, and only slight deviations from the sonic velocity are considered. Under such conditions it is difficult to make accurate velocity and density measurements.

Clearly the present investigation touches on the problem of whether it is possible to have regions of supersonic flow embedded in a subsonic flow without the existence of shock waves. Extensive investigations of this problem based on the inviscid transonic equation have been made and are, for example, discussed by Manwell (1958, 1963), who, with others, concludes that it is not in general possible to obtain smooth inviscid solutions for the transonic flow in such regions. A detailed discussion of this problem in the light of the viscous transonic equation is beyond the scope of the present paper; however, the existence of supersonic regions within regions of subsonic flow does not appear to result in any difficulties when viscous effects are taken into account. This is not a surprising result for the viscous transonic equation inherently contains the possibility of formation of steady shock structures where required by the conditions of the flow, while the inviscid equations do not. Whether, in general, the proper inclusion of viscous effects can eliminate the difficulties encountered by Manwell and others in constructing transonic solutions is certainly a worthwhile subject for future investigation.

There are two basic differences between Szaniawski's (1964*a*, *b*) studies of viscous, transonic nozzle flow and the present work. While Szaniawski permits an arbitrary nozzle contour his solutions are approximate rather than exact as in the present case. Also the expansion scheme used by Szaniawski is different. The nozzle half height is used as the characteristic flow dimension with the result that while $(y/L) \sim O(1)$ the dimensionless co-ordinate corresponding to Y in the present paper has a maximum value of $O(\epsilon^{\frac{1}{2}})$ and $v \sim O(\epsilon^2)$. As a consequence the velocity U is a function of X only in the first-order series solution, and to obtain details of the flow field, in particular a non-trivial result for the shape of the sonic lines, it is necessary to compute the second-order coefficient $u^{(2)}$. In the expansion schemes used here $Y \sim O(1)$ and all details of the flow are recovered from the first-order solution. Szaniawski finds, as in the present paper, that in the Taylor flow viscous effects become crucial near the nozzle throat as the maximum velocity approaches the sonic value.

The author would like to express his appreciation to the Army Research Organization in Durham, N.C., who supported this work under Grant DA-ARO(D)-31-124-G385. The author also would like to thank Mr Thomas David, who carried out many of the computations.

REFERENCES

ACKERET, J., FELDMAN, J. & ROTT, N. 1946 *NACA TM*, no. 1113.

CODDINGTON, E. A. & LEVINSON, N. 1955 *Theory of Ordinary Differential Equations*. New YORK: McGraw Hill.

COLE, J. 1949 Problems in Transonic flow. Ph.D. Thesis, California Institute of Technology.

EMMONS, H. W. 1946 *NACA TN*, no. 1003.

GÖRTLER, H. 1939 *Z. angew. Math. Mech.* **19**, 325.

GUDERLEY, K. G. 1962 *The Theory of Transonic Flow*. Reading: Addison Wesley.

HALL, I. M. & SUTTON, E. P. 1962 *Symposium Transsonicum* (ed. K. Oswatitsch). Berlin: Springer.

HAYES, W. D. 1958 *Fundamentals of Gas Dynamics*, Sec. D. Princeton University Press.

KOPYSTYNSKI, J. & SZANIAWSKI, A. 1965 *Archiwum Mechaniki stosowanej*, **17**, 453.

LIGHTHILL, M. J. 1956 *Surveys in Mechanics* (ed. G. K. Batchelor and R. M. Davies). Cambridge University Press.

LUDFORD, G. S. S. 1951 *J. Aeronaut. Sci.* **18**, 830.

MANWELL, A. R. 1958 *Proc. Roy. Soc.* A, **245**, 481.

MANWELL, A. R. 1963 *Arch. Ratl Mech. Anal.* **12**, 249.

MEYER, T. 1908 Über zweidimensionale Bewegungsvorgänge in einen Gas, das mit Überschallgeschwindigkeit strömt. Ph.D. Dissertation, Göttingen. (See also Carrier, G. 1951 *Foundations of High Speed Aerodynamics*. New York: Dover.)

MINORSKY, N. 1962 *Nonlinear Oscillations*. Princeton: Van Nostrand.

OSWATITSCH, K. & ZIEREP, J. 1960 *Z. angew. Math. Mech.* **40** T, 143.

PEARCY, H. H. 1962 *Symposium Transsonicum* (ed. K. Oswatitsch). Berlin: Springer.

RYZHOV, O. S. & SHEFTER, G. M. 1964 *Prikl. Mat. i Mekh.* **28**, 996.

SICHEL, M. 1963 *Phys. Fluids*, **6**, 653.

SICHEL, M. 1965 *University of Michigan ORA Rep.* no. 05800-1-F.

SZANIAWSKI, A. 1963 *Archiwum Mechaniki stosowanej*, **15**, 904.

SZANIAWSKI, A. 1964*a* Importance des effects de dissipation en écoulement transsonique. *Internat. Council of Aeronaut. Sci. Congress, Paris*, paper no. 64-587.

SZANIAWSKI, A. 1964*b* *Archiwum Mechaniki stosowanej*, **16**, 643.

TAYLOR, G. I. 1910 *Proc. Roy. Soc.* A, **84**, 371.

TAYLOR, G. I. 1930 *ARCR & M*, no. 1381. (See also *The Scientific Papers of G. I. Taylor* (ed. G. K. Batchelor), vol. 3, p. 128. Cambridge University Press.

TOMOTIKA, S. & TAMADA, K. 1950 *Quart. Appl. Math.* **7**, 381.

TOMOTIKA, S. & HASIMOTO, Z. 1950 *J. Math. Phys.* **29**, 105.

WOOD, W. W. 1961 *Phys. Fluids*, **4**, 46.